Quantum fluctuations of position of a mirror in vacuum

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A mirror scattering vacuum fields is submitted to a quantum fluctuating radiation pressure. It also experiences a motional force, related to force fluctuations through fluctuation-dissipation relations. The resulting position fluctuations of the coupled mirror are related to the dissipative part of the mechanical admittance. We compute the time dependent position commutator, which makes apparent the difference between the low-frequency and high-frequency masses, and the anticommutator noise spectrum, which describes the ultimate sensitivity in a length measurement using mirrors.

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INTRODUCTION

A mirror which scatters a laser field is submitted to a fluctuating radiation pressure [1], which is associated through fluctuation-dissipation relations with a motional force, that is a mean force for a moving mirror [2]. The resulting random motion has been studied in detail since it determines the ultimate sensitivity of the interferometers designed for gravitational wave detection [3].

In the absence of laser irradiation, the mirror still scatters vacuum fields, so that the radiation pressure fluctuates [4], and the motional force does not vanish [5,6]. Fluctuations and dissipation are also directly connected in this case [7]. As a consequence of its coupling with vacuum radiation pressure, the mirror's position has to acquire some quantum fluctuations, even if it is introduced as a classical variable. On the other hand, the mirror has proper quantum fluctuations, even if it is not coupled to the field. The purpose of the present paper is to elucidate the relation between those acquired and proper fluctuations.

In order to reach consistent results, it is necessary that the mirror behave as a stable system when coupled to vacuum. For a perfect mirror however, 'runaway solutions' appear, which are analogous to the ones encountered in classical electron theory [8]. We will use the fact that this instability problem is solved [9] by considering a partially transmitting mirror, causal and transparent at frequencies higher than a reflection cutoff ω_C much smaller in reduced units than the mirror's mass m_0

$$\hbar\omega_C \ll m_0 c^2 \tag{1}$$

This condition, which allows one to ignore the recoil effect in the derivation of the motional force, also implies passivity, that is the incapacity of vacuum to sustain runaway solutions. Then, stability results from passivity [9].

We shall obtain that the position fluctuations of the coupled mirror are connected to the dissipative part of the mechanical admittance, in analogy with the thermalized position of an atomic system scattering a thermal field [10], or the thermalized voltage of an electrical oscillator coupled to a resistor at thermal equilibrium [11]. However, at zero temperature, this equilibrium involves vacuum fluctuations rather than thermal ones [12]. The non-commutativity of the force correlation functions will play a central role in this paper, as well as the relation of the commutator with the dissipative part of the motional susceptibility [13].

First, we study the 'input fluctuations', i.e. the fluctuations when coupling is disregarded. The mirror's dynamics are described by two noise spectra corresponding respectively to the commutator and anticommutator, which are related to a mechanical admittance function and which behave resonantly in the vicinity of suspension eigenfrequencies. When coupling to vacuum radiation pressure is taken into consideration, the admittance function is modified. Using the techniques of linear response theory [13], we deduce coupled fluctuations which are related to the modified admittance through fluctuation-dissipation relations. In this derivation, input position fluctuations are considered on an equal foot with input force fluctuations. Yet, it appears that the fluctuations of the coupled variables are completely determined by those of the input force. This means that the results might be obtained from a Langevin equation [14], where all fluctuations are fed by the input force fluctuations; this simpler derivation is presented in Appendix A.

Then, we discuss the two noise spectra related respectively to the commutator and to the anticommutator for the coupled position. The former describes a modification of the canonical commutation relation. Its time dependence, analyzed in Appendix B, appears to be connected to a difference between the low-frequency and high-frequency values of mass. The latter gives the quantum noise on the position of the coupled mirror. When integrated over frequency, it provides the position variance; as a function of time, it describes the quantum

diffusion of a mirror coupled to vacuum radiation pressure. The anticommutator noise spectrum is involved in ultimate quantum limits for a position measurement [3].

We also analyse the autocorrelation of the coupled force and its cross correlation with the coupled position.

For the sake of simplicity, our attention will focus upon the simple problem of an harmonically bound mirror. However, linear response formalism can also be used for the anharmonic oscillator, as shown in Appendix C.

INPUT POSITION FLUCTUATIONS

In a stationary state, we will write any correlation function as the sum of an antisymmetric part, associated with a commutator, and of a symmetric one, related to an anticommutator

$$C_{AA}(t) = \langle A(t)A(0) \rangle - \langle A \rangle^2 = \hbar \left(\sigma_{AA}(t) + \xi_{AA}(t) \right) \quad (2a)$$

$$C_{AA}(t) - C_{AA}(-t) = \langle A(t)A(0) - A(0)A(t) \rangle$$
$$= 2\hbar \xi_{AA}(t) \quad (2b)$$
$$C_{AA}(t) + C_{AA}(-t) = \langle A(t)A(0) - A(0)A(t) \rangle - 2 \langle A \rangle^2$$
$$= 2\hbar \sigma_{AA}(t) \quad (2c)$$

The antisymmetric part ξ_{AA}, typical of quantum fluctuations, would vanish for any classical stochastic treatment.

We will denote for any function f

$$f(t) = \int \frac{\mathrm{d}\omega}{2\pi} f[\omega] e^{-i\omega t}$$

The function $\xi_{AA}[\omega]$ is an odd function of ω while the noise spectrum $\sigma_{AA}[\omega]$ is an even one. If A is a hermitean operator, both are real functions of ω.

Let us first express the position fluctuations of an harmonic oscillator (mass m_0, eigenfrequency ω_0) in its ground state within the linear response formalism. The correlation functions of this 'input position' q^{in} are easily computed (we use shorthand notation where ξ_{qq}^{in} stands for $\xi_{q^{\mathrm{in}}q^{\mathrm{in}}}$)

$$C_{qq}^{\mathrm{in}}(t) = \frac{\hbar}{2m_0\omega_0} \exp(-i\omega_0 t) \quad (3a)$$

$$\xi_{qq}^{\mathrm{in}}(t) = \frac{-i}{2m_0\omega_0} \sin(\omega_0 t) \quad (3b)$$

$$\sigma_{qq}^{\mathrm{in}}(t) = \frac{1}{2m_0\omega_0} \cos(\omega_0 t) \quad (3c)$$

as well as the associated spectra

$$C_{qq}^{\mathrm{in}}[\omega] = \frac{\hbar\pi}{m_0\omega_0} \delta(\omega - \omega_0) \quad (3d)$$

$$\xi_{qq}^{\mathrm{in}}[\omega] = \frac{\pi}{2m_0\omega_0} \left(\delta(\omega - \omega_0) - \delta(\omega + \omega_0) \right) \quad (3e)$$

$$\sigma_{qq}^{\mathrm{in}}[\omega] = \frac{\pi}{2m_0\omega_0} \left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right) \quad (3f)$$

One also defines a mechanical susceptibility χ_{qq}^{in} which describes the linear response of the position to an exerted force [13]

$$\chi_{qq}^{\mathrm{in}}[\omega] = \frac{1}{m_0 \left(\omega_0^2 - (\omega + i\epsilon)^2 \right)} \quad (3g)$$

The parameter $\epsilon \to 0^+$ is inserted in order to ensure the causal character of the susceptibility. The susceptibility is directly connected to the mechanical impedance Z^{in} or to the mechanical admittance Y^{in}

$$-i\omega \chi_{qq}^{\mathrm{in}}[\omega] = Y^{\mathrm{in}}[\omega] = \frac{1}{Z^{\mathrm{in}}[\omega]} \quad (4)$$

Those functions obey the fluctuation-dissipation relations [13] applied to a quantum system at zero temperature

$$C_{qq}^{\mathrm{in}}[\omega] = 2\hbar\theta(\omega)\xi_{qq}^{\mathrm{in}}[\omega] \quad (5a)$$

$$\sigma_{qq}^{\mathrm{in}}[\omega] = \varepsilon(\omega)\xi_{qq}^{\mathrm{in}}[\omega] \qquad \varepsilon(\omega) = \theta(\omega) - \theta(-\omega) \quad (5b)$$

$$2i\xi_{qq}^{\mathrm{in}}[\omega] = \chi_{qq}^{\mathrm{in}}[\omega] - \chi_{qq}^{\mathrm{in}}[-\omega] = 2i \ \mathrm{Im}\, \chi_{qq}^{\mathrm{in}}[\omega] \quad (5c)$$

The $\theta(\omega)$ function may be considered as the limit at zero temperature of Planck spectrum [15]

$$\lim_{T \to 0} \frac{1}{1 - \exp \frac{-\hbar\omega}{k_B T}} = \theta(\omega)$$

Quantities evaluated from the correlation functions at equal times, like the canonical commutator or the variances, may be expressed as integrals over frequency of the spectra $\xi_{qq}[\omega]$ and $\sigma_{qq}[\omega]$. For instance, the commutator of two positions evaluated at different times

$$\langle [q(t), q(0)] \rangle = 2\hbar \xi_{qq}(t)$$

is a real and odd function of t. The commutator between velocity and position is simply deduced ($v(t)$ is the mirror's velocity $q'(t)$)

$$\langle [v(t), q(0)] \rangle = 2\hbar \xi_{vq}(t) = 2\hbar \xi_{qq}'(t) \quad (6)$$

Introducing a specific notation for integrals over frequency

$$\overline{g} = \int \frac{\mathrm{d}\omega}{2\pi} g[\omega] = g(0) \quad (7)$$

the equal-time canonical commutator is

$$\langle [v(0), q(0)] \rangle = 2\hbar \xi_{qq}'(0) = -2i\hbar \overline{\omega \xi_{qq}} \quad (8)$$

The symmetrical correlation function (we assume that the mean value $\langle q \rangle$ vanishes)

$$\langle q(t)q(0) + q(0)q(t) \rangle = 2\hbar \sigma_{qq}(t)$$

is a real and even function of t, and the stationary variances are

$$\Delta q^2 = \left\langle q(0)^2 \right\rangle = \hbar \overline{\sigma_{qq}} \qquad (9a)$$

$$\Delta v^2 = \left\langle v(0)^2 \right\rangle = \hbar \overline{\omega^2 \sigma_{qq}} \qquad (9b)$$

Using equations (3), one recovers the well known expressions

$$\left\langle [v(0), q(0)] \right\rangle = \frac{-i\hbar}{m_0} \qquad (10)$$

$$\Delta q^2 = \frac{\hbar}{2m_0 \omega_0} \qquad \Delta v^2 = \frac{\hbar \omega_0}{2m_0} \qquad (11)$$

The covariance between position and velocity vanishes because $\sigma_{qq}[\omega]$ is an even function of ω.

The foregoing description of quantum fluctuations may seem too sophisticated for such a simple system as an harmonic oscillator. We see in the following that this formalism is well suited to the analysis of fluctuations for the system coupled to vacuum. Relations (5,6,8,9) will still be valid, with different expressions for the noise spectra, and will have a significant content in this case.

Furthermore, expressions (5) for the spectra contain more information than the integrated quantities, even for the uncoupled variables. Consider for instance that position is measured and that the measurement frequency band is characterized by a function $G[\omega]$ with a maximum value of 1 at the frequency of the expected signal. Assuming that quantum noise may be dealt with in the same manner as thermal noise, one gets the effective noise as an integral over the detection bandwidth (with the notation of eq. 7)

$$\Delta q_G^2 = \hbar \overline{G \sigma_{qq}}$$

This effective noise Δq_G^2 may be smaller than the position variance Δq^2. It even vanishes if the suspension eigenfrequencies are outside the detection bandwidth, which is the case in any high sensitivity measurement; in the opposite case, the noise would be of the order of the position variance Δq^2. This explains why the proper quantum fluctuations of the mirror's position may be neglected when analyzing the limits in position measurements. At the end of the paper, we will come back to this discussion and show that the dissipative coupling between mirror and vacuum sets an ultimate quantum limit in position measurements.

INPUT FORCE FLUCTUATIONS

We now consider the input force fluctuations F^{in}, that is the fluctuations of vacuum radiation pressure computed when disregarding reaction of the mirror's motion.

These fluctuations are characterized by stationary correlation functions denoted C_{FF}^{in}, ξ_{FF}^{in} and $\sigma_{FF}^{\mathrm{in}}$, which are connected through fluctuation-dissipation relations to the linear susceptibility χ_{FF}^{in} describing the motional force for small displacements

$$C_{FF}^{\mathrm{in}}[\omega] = 2\hbar \theta(\omega) \xi_{FF}^{\mathrm{in}}[\omega] \qquad (12a)$$

$$\sigma_{FF}^{\mathrm{in}}[\omega] = \varepsilon(\omega) \xi_{FF}^{\mathrm{in}}[\omega] \qquad (12b)$$

$$2i\xi_{FF}^{\mathrm{in}}[\omega] = \chi_{FF}^{\mathrm{in}}[\omega] - \chi_{FF}^{\mathrm{in}}[-\omega] = 2i \ \mathrm{Im}\, \chi_{FF}^{\mathrm{in}}[\omega] \qquad (12c)$$

These relations, which can be regarded as consequences of linear response theory, have been directly checked for the problem of a mirror coupled to vacuum radiation pressure [7,9]. In particular, equations (12a,b) correspond to the fact that vacuum is the equilibrium state at zero temperature, so that radiation pressure can damp the mirror's motion, but cannot excite it.

For a perfect mirror scattering a scalar vacuum field in a two-dimensional (2D) spacetime, the motional force is proportional to the third derivative of the position and corresponds to the following susceptibility, obtained by linearizing the expressions obtained for a perfect mirror in reference [5], or by considering the limit of perfect reflection in reference [7]

$$\chi_{FF}^{\mathrm{in}}[\omega] = \frac{i\hbar \omega^3}{6\pi c^2} = im_0 \tau \omega^3$$

The time constant τ characterizes the weak coupling of mirror to vacuum

$$\tau = \frac{\hbar}{6\pi m_0 c^2}$$

A partially transmitting mirror [7] is described by causal reflectivity and transmittivity functions with a reflection cutoff. This approach finds many justifications. It leads to an expression for the mean Casimir force between two mirrors, which is free from the divergences usually associated with the infiniteness of vacuum energy [16]. It also provides finite and causal expressions for the forces associated with motions of two mirrors [17].

When condition (1) is obeyed, one can ignore the recoil effect in the derivation of the motional force and obtain the susceptibility as

$$\chi_{FF}^{\mathrm{in}}[\omega] = im_0 \tau \omega^3 \Gamma[\omega] \qquad (13a)$$

where $\Gamma[\omega]$ is a cutoff function, whose expression is analysed in detail in reference [9]. A simple example, where the aforementioned conditions are fulfilled, is provided by the following lorentzian model for the reflectivity (r) and transmittivity (s) functions

$$r[\omega] = \frac{-1}{1 - \frac{i\omega}{\Omega}} \qquad s[\omega] = 1 + r[\omega]$$

$$\chi_{FF}^{\mathrm{in}}[\omega] = 6m_0 \tau \Omega^3 \times$$
$$\left(-\frac{i\omega}{\Omega} - \frac{\omega^2}{2\Omega^2} - \left(1 - \frac{i\omega}{\Omega}\right) \ln\left(1 - \frac{i\omega}{\Omega}\right) \right)$$

In the present paper, we will use only some general properties obeyed by the cutoff function and not its explicit expression.

The function $\Gamma[\omega]$ cuts off the high frequencies and is regular at the low frequency limit

$$\Gamma_0 = \Gamma[0] = r_0^2$$

r_0 is the reflection amplitude at $\omega = 0$; $\Gamma_0 = 1$ for a perfectly reflecting mirror at zero frequency. It can be written as the sum of a dissipative and a dispersive parts

$$\Gamma[\omega] = \Gamma_R[\omega] + i\Gamma_I[\omega] \qquad \Gamma_R[\omega] \text{ and } \Gamma_I[\omega] \text{ real}$$
$$\xi_{FF}^{\text{in}}[\omega] = m_0 \tau \omega^3 \Gamma_R[\omega] \tag{13b}$$

The dissipative part Γ_R obeys a positivity property [9]

$$\Gamma_R[\omega] \geq 0 \tag{14}$$

The high frequency behaviour of $\Gamma_R[\omega]$ will be important in the following. Various conditions will be expressed as inequalities for a positive coefficient A

$$A = -\lim_{\omega \to \infty} \frac{\mathrm{d}\ln\Gamma_R}{\mathrm{d}\ln\omega} \tag{15}$$

Using the causal properties of the susceptibility, one can write a dispersion relation with subtractions since the susceptibility does not vanish at high frequencies. As the three quantities $\chi_{FF}^{\text{in}}[0]$, $\chi_{FF}^{\text{in}}{}'[0]$ and $\chi_{FF}^{\text{in}}{}''[0]$ vanish, this relation has a simple form in terms of the cutoff function [9]

$$\Gamma[\omega] = \int \frac{\mathrm{d}\omega'}{i\pi} \frac{\Gamma_R[\omega']}{\omega' - \omega - i\epsilon}$$

It follows that this function behaves as $\frac{1}{\omega}$ at high frequencies

$$-i\omega\Gamma[\omega] \to \omega_C \qquad \text{for } \omega \to \infty \tag{16a}$$
$$\omega_C = \int \frac{\mathrm{d}\omega}{\pi} \Gamma_R[\omega] \tag{16b}$$

Finiteness of the integral (16b) has been assumed, which requires $A > 1$. The parameter ω_C has the dimension of a frequency and may be considered as the definition of the reflection cutoff if $\Gamma_0 = 1$; otherwise, it is the product of Γ_0 and of the cutoff.

In order to deal with stationary fluctuations, we will require that the mirror coupled to vacuum behaves as a stable system. Runaway solutions occur for a perfect mirror $\omega_C \to \infty$, but not for a mirror such that [9]

$$\omega_C \tau \ll 1 \tag{17}$$

Indeed, the coupled system obeys passivity [18] in this case. This property is connected to the fact that vacuum cannot deliver energy for sustaining runaway solutions, so that stability follows from passivity [9]. Then, a partially transmitting mirror obeying equation (1) is stable.

FLUCTUATIONS OF COUPLED VARIABLES

We now come to the evaluation of coupled position and force fluctuations.

We describe the coupling by two linear response equations

$$F[\omega] = F^{\text{in}}[\omega] + \chi_{FF}^{\text{in}}[\omega]q[\omega] \tag{18a}$$
$$q[\omega] = q^{\text{in}}[\omega] + \chi_{qq}^{\text{in}}[\omega]F[\omega] \tag{18b}$$

Each coupled variable, F or q, is the sum of the corresponding input quantity, F^{in} or q^{in}, and of a linear response to the other variable, q or F, characterized by the linear susceptibilities χ_{FF}^{in} and χ_{qq}^{in} which have been previously discussed.

The linear equations (18) are easily solved, leading to expressions of the coupled variables in terms of the uncoupled ones and of linear susceptibilities which are denoted χ_{qq}, χ_{FF}, $\chi_{qF} = \chi_{Fq}$ for reasons which will soon become apparent

$$q[\omega] = \chi_{qq}[\omega]F^{\text{in}}[\omega] + \chi_{Fq}[\omega]q^{\text{in}}[\omega] \tag{19a}$$
$$F[\omega] = \chi_{Fq}[\omega]F^{\text{in}}[\omega] + \chi_{FF}[\omega]q^{\text{in}}[\omega] \tag{19b}$$

$$\chi_{qq}[\omega] = \frac{1}{\frac{1}{\chi_{qq}^{\text{in}}[\omega]} - \chi_{FF}^{\text{in}}[\omega]} \tag{19c}$$

$$\chi_{Fq}[\omega] = \frac{1}{1 - \chi_{FF}^{\text{in}}[\omega]\chi_{qq}^{\text{in}}[\omega]} \tag{19d}$$

$$\chi_{FF}[\omega] = \frac{1}{\frac{1}{\chi_{FF}^{\text{in}}[\omega]} - \chi_{qq}^{\text{in}}[\omega]} \tag{19e}$$

One then deduces the commutators for the coupled operators ($\xi_{qF} = \xi_{Fq}$; $\xi_{qF}^{\text{in}} = \xi_{Fq}^{\text{in}} = 0$)

$$\xi_{qq}[\omega] = \chi_{qq}[\omega]\xi_{FF}^{\text{in}}[\omega]\chi_{qq}[-\omega]$$
$$+ \chi_{Fq}[\omega]\xi_{qq}^{\text{in}}[\omega]\chi_{Fq}[-\omega] \tag{20a}$$

$$\xi_{Fq}[\omega] = \chi_{qq}[\omega]\xi_{FF}^{\text{in}}[\omega]\chi_{Fq}[-\omega]$$
$$+ \chi_{Fq}[\omega]\xi_{qq}^{\text{in}}[\omega]\chi_{FF}[-\omega] \tag{20b}$$

$$\xi_{FF}[\omega] = \chi_{Fq}[\omega]\xi_{FF}^{\text{in}}[\omega]\chi_{Fq}[-\omega]$$
$$+ \chi_{FF}[\omega]\xi_{qq}^{\text{in}}[\omega]\chi_{FF}[-\omega] \tag{20c}$$

Using the fluctuation-dissipation relations (5,12) for the input variables, one derives, after straighforward transformations, similar relations for the coupled ones (A and B stand for q or F)

$$2i\xi_{AB}[\omega] = \chi_{AB}[\omega] - \chi_{AB}[-\omega] \tag{21a}$$

One also obtains, after similar transformations

$$C_{AB}[\omega] = 2\hbar\theta(\omega)\xi_{AB}[\omega] \tag{21b}$$
$$\sigma_{AB}[\omega] = \varepsilon(\omega)\xi_{AB}[\omega] \tag{21c}$$

These expressions constitute a consistent description of the coupled fluctuations. The input fluctuations q^{in} and F^{in} have been treated on an equal foot in the derivation and the resulting expressions are symmetrical in the two quantities q and F. The susceptibility functions χ_{qq}, χ_{FF}

and χ_{Fq} play the same role as χ_{qq}^{in} and χ_{FF}^{in} and are associated to coupled variables rather than to input ones.

We show in Appendix A that these expressions can be rewritten in a simpler, but less symmetric, form where the input position fluctuations q^{in} no longer appear. The basic reason is that the two functions χ_{Fq} and χ_{FF} vanish at the suspension eigenfrequencies $\pm\omega_0$ where the input position fluctuations q^{in} are concentrated. Then, equations (19) become

$$q[\omega] = \chi_{qq}[\omega]F^{\mathrm{in}}[\omega] \tag{19'a}$$

$$F[\omega] = \chi_{Fq}[\omega]F^{\mathrm{in}}[\omega] \tag{19'b}$$

$$\chi_{qq}[\omega] = \frac{1}{m_0\left(\omega_0^2 - \omega^2\right) - \chi_{FF}^{\mathrm{in}}[\omega]} \tag{19'c}$$

$$\chi_{Fq}[\omega] = m_0\left(\omega_0^2 - \omega^2\right)\chi_{qq}[\omega] \tag{19'd}$$

Also, the commutators (20) are simplified to

$$\begin{aligned}\xi_{qq}[\omega] &= \chi_{qq}[\omega]\xi_{FF}^{\mathrm{in}}[\omega]\chi_{qq}[-\omega] \\ &= |\chi_{qq}[\omega]|^2\xi_{FF}^{\mathrm{in}}[\omega]\end{aligned} \tag{20'a}$$

$$\begin{aligned}\xi_{Fq}[\omega] &= \chi_{qq}[\omega]\xi_{FF}^{\mathrm{in}}[\omega]\chi_{Fq}[-\omega] \\ &= m_0\left(\omega_0^2 - \omega^2\right)\xi_{qq}[\omega]\end{aligned} \tag{20'b}$$

$$\begin{aligned}\xi_{FF}[\omega] &= \chi_{Fq}[\omega]\xi_{FF}^{\mathrm{in}}[\omega]\chi_{Fq}[-\omega] \\ &= m_0^2\left(\omega_0^2 - \omega^2\right)^2\xi_{qq}[\omega]\end{aligned} \tag{20'c}$$

In other words, the position fluctuations of the coupled mirror, as well as the coupled force fluctuations, are entirely determined by input fluctuations F^{in} of vacuum radiation pressure. Although q^{in} and F^{in} have been treated on an equal foot, a privileged role is attributed to F^{in} in the end. As usual in the theory of irreversible phenomena, the reason for this asymmetry is that F^{in} corresponds to a dense spectrum whereas q^{in} corresponds to a discrete one. As a result, the linear response equations (18) can be replaced by a simple Langevin equation (see Appendix A).

This suggests that the present approach presents some analogy with the stochastic interpretations of quantum mechanics, like stochastic electrodynamics [19] or stochastic mechanics [20]. It indeed suggests that quantum fluctuations of mirror's position can effectively be considered as a consequence of its coupling with zero point fields. As a result, these fluctuations scale as $\frac{\hbar}{m_0}$ with $\hbar$ introduced through vacuum fluctuations and m_0 introduced as the effective mass in the Newton equation of motion. It has however to be stressed that the non-commutative character of the vacuum fluctuations plays a central role in our derivation, whereas the zero point fields are often treated as classical variables in stochastic interpretations.

Moreover, it has to be noted that the reverse situation could occur where the field fluctuations would be dominated by position ones. Consider for example the original Casimir problem [21] with two perfect mirrors. In this case, the intracavity field possesses a discrete spectrum. Then, one concludes that the outside field fluctuations determine the position fluctuations of the mirrors, turning their discrete spectrum into a continuous one, which themselves determine the intracavity field fluctuations.

COMMUTATION RELATIONS FOR MIRROR'S VARIABLES AND EFFECTIVE MASS

From fluctuation-dissipation relations (21), the position commutator ξ_{qq} of the coupled mirror is directly connected to the mechanical admittance Y and impedance Z (compare with the similar relation 4 for the uncoupled mirror; Y_R and Z_R are the real parts, i.e. the dissipative parts, of Y and Z; a discussion of position fluctuations in terms of Y_R and Z_R is given in Appendix A)

$$-i\omega\chi_{qq}[\omega] = Y[\omega] = \frac{1}{Z[\omega]} \tag{22a}$$

$$\omega\xi_{qq}[\omega] = Y_R[\omega] = \frac{Z_R[\omega]}{|Z[\omega]|^2} \tag{22b}$$

In the following, we rewrite the susceptibility as (see eqs 13,19'c)

$$\frac{1}{\chi_{qq}[\omega]} = m_0\omega_0^2 - \omega^2 m[\omega] \tag{23a}$$

$$m[\omega] = m_0\left(1 + i\omega\tau\Gamma[\omega]\right) \tag{23b}$$

The coefficient m_0 appears as the quasistatic mass, that is the effective mass at the low frequency limit. Using the relations (16) derived from the causal properties of the susceptibility, one deduces a different value for the high-frequency mass m_∞

$$m_\infty = m[\infty] = m_0 - \mu \tag{23c}$$

$$\mu = m_0\omega_C\tau = \frac{\hbar\omega_C}{6\pi c^2} \tag{23d}$$

μ appears as an 'induced mass', always positive, and is similar to the 'electromagnetic mass' of classical electron theory [8] (using eqs 16b and 13b)

$$\mu = \int\frac{\mathrm{d}\omega}{\pi}\frac{\xi_{FF}^{\mathrm{in}}[\omega]}{\omega^3} \tag{23e}$$

Here, the induced mass μ is much smaller than the quasistatic mass m_0 for mirrors obeying equation (1), and low- and high-frequency values of the mass only slightly differ.

Using equations (6,22), one deduces that the canonical commutator between velocity and position is directly connected to the admittance function, as it is modified by coupling to vacuum (the uncoupled expression was the same with Y^{in} in place of Y; see eq. 4)

$$\xi_{vq}[\omega] = -i\omega\xi_{qq}[\omega] = -\frac{i}{2}\left(Y[\omega] + Y[-\omega]\right) \tag{24a}$$

$$\langle[v(t),q(0)]\rangle = -i\hbar\left(Y(t) + Y(-t)\right) \tag{24b}$$

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The time dependence of the function $Y(t)$ is analysed in Appendix B. Expanding in the small parameters $\omega_0 \tau$ and $\omega_C \tau$, assuming that ω_0 is much smaller than ω_C, we get the approximate expression

$$Y(t) = Y^{\text{in}}(t) \exp \frac{-\gamma t}{2} + \Delta Y(t)$$

The first term corresponds to the uncoupled admittance, which is damped on very long times with a damping constant

$$\gamma \approx \omega_0^2 \tau \Gamma_0$$

The second part $\Delta Y(t)$ is a 'bump' centered at $t = 0$, with a small height $\frac{\omega_C \tau}{m_0}$ and a small width of the order of $\frac{\Gamma_0}{\omega_C}$, that is the inverse of the cutoff frequency (see eqs 16).

The various frequencies involved in the admittance function scale as follows

$$\gamma \ll \omega_0 \ll \omega_C \ll \frac{1}{\tau} \tag{25a}$$

$$\frac{\gamma}{\omega_0} \approx \omega_0 \tau \Gamma_0 \tag{25b}$$

Then, the time $\frac{1}{\gamma}$ is much longer than any other characteristic time

$$\tau \ll \frac{1}{\omega_C} \ll \frac{1}{\omega_0} \ll \frac{1}{\gamma} \tag{25c}$$

One deduces (see Appendix B) that the commutator has the following values as a function of time

$$\langle [v(t), q(0)] \rangle = -\frac{i\hbar}{m_\infty} \qquad \text{for } t \ll \frac{1}{\omega_C} \tag{26a}$$

$$= -\frac{i\hbar}{m_0} \qquad \text{for } \frac{1}{\omega_C} \ll t \ll \frac{1}{\omega_0} \tag{26b}$$

$$= -\frac{i\hbar}{m_0} \cos(\omega_0 t) \exp\left(-\frac{\gamma |t|}{2}\right)$$

$$\text{for } \frac{1}{\omega_C} \ll t \tag{26c}$$

The commutator at times shorter than the oscillation period, in particular, the commutator for an unbound mirror $\omega_0 = 0$, has a time dependence directly connected to the difference between the two masses which appear in equations (23). At short times $t \ll \frac{1}{\omega_C}$ and longer times $\frac{1}{\omega_C} \ll t$ respectively, the commutator is related to the high frequency mass m_∞ (mass for $\omega_C \ll \omega$) and to the quasistatic mass m_0 (mass for $\omega \ll \omega_C$).

This discussion suggests that m_∞ has to be considered as the bare mass while m_0 contains a part due to coupling with vacuum. In this spirit, we may rewrite equations (23) as

$$m[\omega] = m_\infty + \mu[\omega] \tag{23'a}$$

$$\mu[\omega] = m_0 \tau (\omega_C + i\omega \Gamma[\omega]) \tag{23'b}$$

$$\mu[0] = m_0 \omega_C \tau = \mu \qquad \mu[\infty] = 0 \tag{23'c}$$

where the frequency dependent function $\mu[\omega]$ appears as a contribution of the vacuum energy bound to the mirror, and swept along by its motion. This phenomenon is effective at low frequencies, but not at high frequencies since the mirror is transparent at this limit and its motion is decoupled from vacuum fields.

POSITION AND VELOCITY VARIANCES

We now come to the evaluation of the position and velocity variances which can be deduced as integrals of the symmetrical correlation function (see eqs 9).

Before evaluating these integrals, we derive a Schwartz inequality for the product of the two variances

$$\overline{(1 + \alpha|\omega|)^2 \sigma_{qq}} \geq 0 \qquad \text{for any } \alpha \text{ real}$$

$$\overline{\sigma_{qq}}\,\overline{\omega^2 \sigma_{qq}} \geq \left(\overline{|\omega|\sigma_{qq}}\right)^2$$

Using equations (21c) and (24), one gets

$$\sigma_{qq}[\omega] = \varepsilon(\omega)\xi_{qq}[\omega] \qquad \overline{|\omega|\sigma_{qq}} = \overline{\omega\xi_{qq}} = i\xi_{vq}(0)$$

leading to a Heisenberg inequality, which sets a lower bound on the product of variances, related to the equal-time canonical commutator

$$\Delta q \Delta v \geq \frac{\hbar}{2m_\infty} \geq \frac{\hbar}{2m_0} \tag{27}$$

In order to explicitly evaluate the variances, we write the noise spectra as (see eqs 20')

$$\xi_{qq}[\omega] = \frac{1}{m_0} \frac{\omega^3 \tau \Gamma_R}{\left(\omega^2 - \omega_0^2 - \omega^3 \tau \Gamma_I\right)^2 + \left(\omega^3 \tau \Gamma_R\right)^2} \tag{28a}$$

$$\sigma_{qq}[\omega] = \varepsilon(\omega)\xi_{qq}[\omega] = |\xi_{qq}[\omega]| \tag{28b}$$

As τ is a very small time, the noise spectrum σ_{qq} is approximately a sum of two narrow peaks around the two suspension eigenfrequencies. The width of the peaks is the damping constant γ encountered previously in the time dependence of the commutator. Besides these peaks, the noise spectrum also contains a small background (see Appendix B). If one considers the limit where the mirror is decoupled from vacuum, one gets a vanishing width γ for the peaks and a vanishing height for the background (compare with the discussion of the time dependence of $Y(t)$). Then, a simple expression is obtained, which corresponds to the uncoupled oscillator (see eqs 3), so that the usual variances (11) are recovered. However, this limit must be considered more carefully, as shown by an examination of the convergence of the integrals (9).

Due to the high frequency behaviour of $\Gamma_R[\omega]$ (see eq. 15), a finite value is obtained for Δq^2 as soon as $A > 0$ ($A > 1$ was already needed for getting a finite value for ω_C). It can be checked that the value for Δq^2 is close to the expression (11) associated with an uncoupled

oscillator ($\tau \to 0$ or $\Gamma_0 \to 0$). A more stringent condition $A > 2$ is required in order to get a finite value for Δv^2; for $A = 2$, a logarithmic divergence is obtained. This implies that the velocity variance may be completely changed by the coupling, even for a vanishing value of τ or Γ_0. The fact that we may obtain an infinite variance for the instantaneous velocity is not surprising, a similar result being obtained in classical theory of Brownian motion.

In the particular case of an unbound mirror ($\omega_0 = 0$) coupled to vacuum, one gets

$$\xi_{qq}[\omega] = \frac{1}{m_0 \omega} \frac{\tau \Gamma_R}{\left(1 - \omega \tau \Gamma_I\right)^2 + \left(\omega \tau \Gamma_R\right)^2} \tag{29a}$$

$$\sigma_{qq}[\omega] = |\xi_{qq}[\omega]| \tag{29b}$$

The low frequency divergence of the spectra (29) leads to an infinite position variance Δq^2, typical of the unbound diffusion process experienced by a free particle submitted to a random force. For a mirror in vacuum, it could be expected that the stationary state is, as in standard quantum mechanics, a velocity eigenstate ($\Delta v^2 = 0$; $\Delta q^2 = \infty$). However, the integral Δv^2 diverges if $A \leq 2$, as in the general case of a suspended mirror's motion.

QUANTUM DIFFUSION OF A MIRROR COUPLED TO VACUUM

The quantum diffusion of a mirror coupled to vacuum may be characterized quantitatively by the function

$$\Delta(t) = \frac{1}{2} \left\langle \left(q(t) - q(0)\right)^2 \right\rangle = \hbar \left(\sigma_{qq}(0) - \sigma_{qq}(t)\right)$$
$$= \int \frac{d\omega}{2\pi} \hbar \sigma_{qq}[\omega] \left(1 - \cos\left(\omega t\right)\right)$$

This integral is well defined for any value of t, even for an unbound mirror ($\omega_0 = 0$), and it vanishes when t goes to zero. This is also the case for the integral associated with $\Delta'(t)$ (we suppose $A > 1$), but not for $\Delta''(t)$, which is not defined if $A \leq 2$ ($\Delta''(0) = \Delta v^2$).

Interesting results about quantum diffusion are obtained by using the analytic properties of the correlation function $C_{qq}(t)$. Since the noise spectrum $C_{qq}[\omega]$ contains only positive frequency components (see eq. 21b), $C_{qq}(t)$ is analytic and regular in the half plane $\text{Im}\, t < 0$ [22,23]. Then, $\sigma_{qq}(t)$ may be deduced from $\xi_{qq}(t)$ through a dispersion relation ($\mathcal{P}$ stands for a principal value)

$$\sigma_{qq}(t) = \int dt' \xi_{qq}(t - t') \mathcal{P} \frac{1}{i\pi t'} \tag{30}$$

One obtains a simple integral relation between the time dependent commutator and the position variance by setting $t = 0$ in the preceding integral. One also obtains by deriving the expression of $\sigma_{qq}(t)$

$$\sigma'_{qq}(t) = \int dt' \xi'_{qq}(t - t') \mathcal{P} \frac{1}{i\pi t'}$$

In the particular case of an unbound mirror, ξ'_{qq} may be written (see Appendix B)

$$\xi'_{qq}(t) = -i \left(\frac{1}{2m_0} + \Delta Y_R(t) \right)$$

The contribution to the integral $\sigma'_{qq}(t)$ of the first term, which is time independent, vanishes. Since ΔY_R differs from zero only at short times, of the order of $\frac{1}{\omega_C}$, the asymptotic behaviour at long times of $\sigma'_{qq}(t)$ can be evaluated as

$$\sigma'_{qq}(t) \approx -\frac{\Delta Y_R[0]}{\pi t} \approx -\frac{\tau \Gamma_0}{\pi m_0 t} \qquad \text{for } \omega_C t \gg 1$$

It follows that the quantum diffusion of an unbound mirror coupled to vacuum is characterized by a logarithmic behaviour at long times [24]

$$\Delta(t) \approx \frac{\hbar \tau \Gamma_0}{\pi m_0} \ln \frac{t}{t_0} \qquad \text{for } \omega_C t \gg 1$$

t_0 is an integration constant, of the order of $\frac{1}{\omega_C}$. This diffusion corresponds to very small displacements, the length scale being in fact the Compton wavelength $\frac{\hbar}{m_0 c}$ associated with the mirror

$$\Delta(t) \approx \frac{\Gamma_0}{6\pi^2} \left(\frac{\hbar}{m_0 c} \right)^2 \ln \left(\omega_C t \right) \tag{31}$$

It has to be noted that thermal effects [15] change the behaviour of $\Delta(t)$ at long times, even at a low temperature.

FLUCTUATIONS OF COUPLED FORCE

It is interesting to discuss the autocorrelation of the coupled force F, that is the force computed in presence of the mirror's radiation reaction, and its correlation with the coupled position.

The autocorrelation of the coupled force F is given by equation (20'c); ξ_{FF} is identical to ξ_{FF}^{in}, except in the narrow resonance peaks where the fluctuations of the coupled force are smaller than those of the input force (compare with eqs 28)

$$\xi_{FF}[\omega] = \xi_{FF}^{\text{in}}[\omega] \frac{\left(\omega^2 - \omega_0^2\right)^2}{\left(\omega^2 - \omega_0^2 - \omega^3 \tau \Gamma_I\right)^2 + \left(\omega^3 \tau \Gamma_R\right)^2} \tag{32}$$

The correlation of the coupled force with the coupled position is described by equation (20'b). This expression implies that coupled position and coupled force evaluated at equal times commute; $\xi_{Fq}[\omega]$ is an odd function of ω as $\xi_{qq}[\omega]$, so that

$$\xi_{Fq}(0) = 0 \tag{33}$$

From the same argument of parity, it follows that coupled velocity and coupled force do not necessarily commute when evaluated at equal time.

One also deduces the symmetrical correlation function between q and F

$$\sigma_{Fq}[\omega] = \varepsilon(\omega)\xi_{Fq}[\omega] = m_0\left(\omega_0^2 - \omega^2\right)\sigma_{qq}[\omega]$$
$$\langle q(0)F(0)\rangle = \langle F(0)q(0)\rangle = \hbar\overline{\sigma_{Fq}}$$
$$= m_0\left(\omega_0^2\Delta q^2 - \Delta v^2\right) \tag{34}$$

which is reminiscent of the virial theorem [25].

It appears from this discussion that the position of the mirror and the vacuum pressure exerted upon it are intimately correlated: at each frequency, they are directly proportional to each other. Therefore the quantum fluctuations of the mirror cannot be treated independently of those of vacuum radiation pressure.

ULTIMATE QUANTUM LIMITS IN A POSITION MEASUREMENT

We show in this section that the noise spectrum σ_{qq} can be considered as the ultimate quantum limit for the sensitivity in a position measurement, when mirrors coupled to vacuum are used. More precisely, we discuss how this limit may be reached, at least in principle; we will not analyse in detail the effective realization of this sensitivity.

We consider a simple model of position measurement where a laser is sent onto the mirror and where the phase Φ of the reflected beam is monitored [3]. This phase Φ is related to the mirror's position q through (k_0 is the laser wavenumber)

$$\Phi(t) = 2k_0\left(q(t) + \delta q_{RP}(t)\right) + \delta\Phi(t)$$

Here, $q(t)$ represents the position discussed in foregoing sections, with its quantum fluctuations; $\delta\Phi$ represents the quantum fluctuations of the phase of the incident field, and δq_{RP} stands for the random motion of the mirror due to the radiation pressure δF_{RP} exerted upon it [1], proportional to the fluctuations δI of laser intensity

$$\delta F_{RP}(t) = 2\hbar k_0 \delta I(t)$$

In the following, we will consider that the fluctuations of q, which can be assigned to vacuum radiation pressure (the vacuum force at frequency ω is associated with vacuum field fluctuations at frequencies comprised between 0 and ω), are not correlated with the fluctuations of δq_{RP} and $\delta\Phi$ (field fluctuations at frequencies close to the laser frequency). We will also assume that the field fluctuations around the laser frequency can be transformed (squeezed) at our convenience, while the lower frequencies vacuum fluctuations are not modified.

In a linear analysis, the response of the mirror to radiation pressure is described by the susceptibility function

$$\delta q_{RP}[\omega] = \chi_{qq}[\omega]\delta F_{RP}[\omega]$$

The estimator $\widehat{q}[\omega]$ for a frequency component $q[\omega]$ of position can be written

$$\widehat{q}[\omega] = \frac{\Phi[\omega]}{2k_0} = q[\omega] + q_N[\omega]$$

The added noise $q_N[\omega]$ is a sum of two terms associated respectively with phase and intensity fluctuations (compare with eqs 11 of ref. [3])

$$q_N[\omega] = \frac{\delta\Phi[\omega]}{2k_0} + 2\hbar k_0\chi_{qq}[\omega]\delta I[\omega]$$

When phase and intensity fluctuations are considered as uncorrelated, the corresponding noise spectrum is obtained as a sum of two contributions [1]

$$\sigma_{q_N q_N}[\omega] = \frac{\sigma_{\Phi\Phi}[\omega]}{\left(2k_0\right)^2} + \left(2\hbar k_0\right)^2\left|\chi_{qq}[\omega]\right|^2\sigma_{II}[\omega]$$

The minimum value reached by this expression when the various parameters are varied, with variations of $\sigma_{\Phi\Phi}$ and σ_{II} constrained by a Heisenberg inequality, is the standard quantum limit

$$\sigma_{q_N q_N}^{\text{SQL}}[\omega] = |\chi_{qq}[\omega]|$$

It is known that the sensitivity can be pushed beyond the standard quantum limit [2]. Indeed, phase and intensity fluctuations are linearly superimposed in the fluctuations of the monitored signal, which therefore depend also upon the correlation $\sigma_{I\Phi}$. Consequently, one can reduce noise by squeezing a well chosen quadrature component of the fields. A lower bound σ^{UQL} is obtained by optimising the parameters characterizing the squeezed fields, the variations of $\sigma_{\Phi\Phi}$, σ_{II} and $\sigma_{I\Phi}$ being constrained by a Heisenberg inequality. It is determined by the dissipative part of the mirror's susceptibility [3]. With notations of the present paper, this 'ultimate quantum limit' can be written

$$\sigma_{q_N q_N}^{\text{UQL}}[\omega] = |\xi_{qq}[\omega]| = \sigma_{qq}[\omega]$$

We have used relation (28b) between the noise spectrum and the dissipative part of the susceptibility.

It is instructive to measure the fluctuations of the estimated position $\widehat{q}$ as a noise energy N per unit bandwidth (fluctuations of q and q_N are uncorrelated)

$$N[\omega] = \frac{m_0}{2}\left(\omega_0^2 + \omega^2\right)\left(C_{qq}[\omega] + C_{q_N q_N}[\omega]\right)$$

The standard quantum limit corresponds to

$$N^{\text{SQL}}[\omega] = \hbar\theta(\omega)m_0 \times$$
$$\left(\omega_0^2 + \omega^2\right)\left(|\xi_{qq}[\omega]| + |\chi_{qq}[\omega]|\right) \tag{35a}$$
$$\approx \hbar\theta(\omega) \qquad \text{for } \omega_0 \ll \omega \ll \omega_S \tag{35b}$$

This limit is entirely determined by added fluctuations (second term in 35a) and proper quantum fluctuations of

position (first term in 35a) do not contribute (these fluctuations are concentrated in a very narrow peak at the suspension eigenfrequency). This confirms that proper fluctuations can be forgotten when analyzing a position measurement with a sensitivity level around standard quantum limit.

In contrast, the ultimate quantum limit reaches the level set by the dissipative character of coupling with vacuum; proper fluctuations have the same contribution as added fluctuations

$$N^{\mathrm{UQL}}[\omega] = \hbar\theta(\omega)m_0 \left(\omega_0^2 + \omega^2\right) 2|\xi_{qq}[\omega]| \qquad (36\text{a})$$
$$\approx 2\hbar\theta(\omega)\omega\tau\Gamma_0 \qquad \text{for } \omega_0 \ll \omega \ll \omega_S \quad (36\text{b})$$

This limit (36) is far beyond the standard quantum limit (35). It has to be stressed however that the input state of the field used in the measurement must be carefully controlled, in the frequency band around laser frequency which is involved in the measurement. Reaching effectively the ultimate limit in a real experiment may be a rather difficult task.

This is made clear by expressing this limit as a position variance

$$\Delta\widehat{q}^2 \approx \frac{2B}{\omega}\frac{2\Gamma_0}{3\pi}\left(\frac{\hbar}{m_0 c}\right)^2$$
$$\text{for } \omega_0 \ll \omega - B, \omega + B \ll \omega_S$$

where $\frac{\hbar}{m_0 c}$ is the Compton wavelength of the mirror and $2B$ the measurement bandwidth (signal measured through a filter characterized by a function $G[\omega]$ having a maximum value of 1 at expected signal frequency)

$$2B = \overline{G}$$

The effective noise $\Delta\widehat{q}^2$ in the measurement may be much smaller than the position variance: the integral of a positive quantity over a limited bandwidth is obviously smaller than the integral of the same quantity over all frequencies (see eqs 9). This occurs, if the input state of the laser is optimized, when the suspension eigenfrequencies are outside the detection bandwidth; in the opposite case, the effective noise would be of the order of the position variance. Incidentally, this discussion means that quantum noise may be reduced by a narrow band detection, exactly in the same manner as thermal noise [26].

DISCUSSION

Any mirror, more generally any scatterer, is coupled to vacuum radiation pressure, so that its position acquires quantum fluctuations from vacuum. This mechanism is quite analogous to the thermalization of the variables of a system coupled to a thermal bath, but it involves vacuum (i.e. zero temperature) fluctuations.

Quite generally, coupling a classical quantity to quantum ones leads to inconsistencies [27]. In contrast, generic quantum properties are preserved in a consistent treatment, as a consequence of fluctuation-dissipation relations. Then, position fluctuations of the coupled mirror are connected to the dissipative part of its mechanical admittance.

It appears that fluctuations of the coupled variables are determined by input fluctuations corresponding to a continuous spectrum (i.e. the 'dissipative' ones), which are here the force fluctuations. It can therefore be asserted that quantum properties of position are generated from vacuum fluctuations: the same results would be obtained if the position were initially treated as a classical number. Nevertheless, the proper position fluctuations are recovered in the end, in the limit of decoupling. Coupling with vacuum is very weak, making the value of τ extremely small, and this justifies that usual quantum mechanics can provide us with a good description of position fluctuations.

The canonical commutation relation between position and velocity is slightly modified, this modification being only effective at small times of the order of the reflection delay. This time dependence is connected to the difference between the bare mass (at high frequencies where field energy cannot follow the mirror's motion) and the quasistatic mass (at low frequencies where a finite part of the vacuum energy is swept along the mirror's motion).

The noise spectrum, which describes the quantum fluctuations of position for a mirror coupled to vacuum, consists in two narrow resonance peaks close to the suspension eigenfrequencies and a small broad background. When the peaks are approximated as Dirac distributions and the background disregarded, integrals over frequency of the noise spectrum reproduce the dispersions Δq and Δv expected for a quantum harmonic oscillator in its ground state.

However, differences exist which may have some importance. Although finite in the uncoupled case, the velocity dispersion may be infinite for the coupled oscillator, a property reminiscent of classical theory of Brownian motion. Also, the resonance peaks have a small width, typical of dissipative coupling with vacuum, so that the correlation functions are damped on very long times and differ from the usual dissipationless expressions. For an unbound mirror coupled to vacuum, a logarithmic diffusion is obtained (no diffusion for an unbound uncoupled mirror), the length scale of which is given by the Compton wavelength associated with the mirror.

Finally, the proper quantum fluctuations of mirrors used in a length measurement such as interferometric detection of gravitational waves are accounted for as soon as coupling with vacuum radiation pressure is treated. The noise spectrum associated with position sets the ultimate bound on sensitivity when the measurement is optimized. Then, the effective noise in the measurement can be much smaller than the position variance computed from quantum mechanics, as soon as the detection bandwidth does not contain the suspension eigenfrequencies. Quantum noise, as other sources of noise, may be reduced

by monitoring a signal away from resonance frequencies.

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APPENDIX A: ELIMINATION OF THE INPUT POSITION FLUCTUATIONS

The linear response equations (18) can be replaced by simpler, but less symmetric, ones where the input position fluctuations q^{in} no longer appear.

The demonstration is based upon the following relations (deduced from eqs 19)

$$\chi_{Fq}[\omega] = \frac{\chi_{qq}[\omega]}{\chi_{qq}^{\mathrm{in}}[\omega]} = m_0 \left(\omega_0^2 - \omega^2\right) \chi_{qq}[\omega]$$

$$\chi_{FF}[\omega] = \chi_{Fq}[\omega]\chi_{FF}^{\mathrm{in}}[\omega] = m_0 \left(\omega_0^2 - \omega^2\right) \chi_{qq}[\omega]\chi_{FF}^{\mathrm{in}}[\omega]$$

It follows that the functions χ_{Fq} and χ_{FF} vanish at the suspension eigenfrequencies

$$\chi_{Fq}[\pm\omega_0] = \chi_{FF}[\pm\omega_0] = 0$$

Since the input position fluctuations q^{in} are concentrated at these frequencies (see eqs 3), they cannot feed the fluctuations of the coupled variables which are therefore entirely determined by the input force fluctuations. Equations (19) become

$$q[\omega] = \chi_{qq}[\omega]F^{\mathrm{in}}[\omega]$$

$$F[\omega] = m_0 \left(\omega_0^2 - \omega^2\right) q[\omega]$$

$$\chi_{qq}[\omega] = \frac{1}{m_0 \left(\omega_0^2 - \omega^2\right) - \chi_{FF}^{\mathrm{in}}[\omega]}$$

Although q^{in} and F^{in} have been treated on an equal foot, F^{in} plays a dominant role in the end. This is due to the property

$$\frac{\xi_{qq}^{\mathrm{in}}[\omega]}{\chi_{qq}^{\mathrm{in}}[\omega]} = 0$$

which is satisfied for the discrete spectrum of equations (3). The same property is not fulfilled by the dense spectrum associated with the force (see eq. 13). It results that expressions (20) of the commutators can be rewritten as (20').

A mirror coupled to vacuum then obeys a simple equation of motion where, as in Langevin theory of Brownian motion [14], the force F is a sum of a Langevin force F^{in}, the input force, and of a long term cumulative effect, the motional force,

$$m_0\omega_0^2 q(t) + m_0 q''(t) = F(t)$$

$$m_0 \left(\omega_0^2 - \omega^2\right) q[\omega] = F[\omega]$$

$$F[\omega] = F^{\mathrm{in}}[\omega] + \chi_{FF}^{\mathrm{in}}[\omega]q[\omega]$$

These equations can be obtained directly by setting $q^{\mathrm{in}} = 0$ in (18).

These equations can be solved in terms of a mechanical impedance Z or of a mechanical admittance Y, which connect the applied force and the effective velocity

$$F^{\mathrm{in}}[\omega] = Z[\omega]v[\omega]$$

$$Z[\omega] = \frac{m_0\omega_0^2}{-i\omega} - i\omega m_0 \left(1 + i\omega\tau\Gamma[\omega]\right)$$

$$v[\omega] = Y[\omega]F^{\mathrm{in}}[\omega]$$

$$Y[\omega] = \frac{1}{Z[\omega]} = -i\omega\chi_{qq}[\omega]$$

The dissipative parts of the impedance and admittance functions, defined as their real parts Z_R and Y_R in the frequency domain, are positive at all frequencies (see eq 14)

$$Z_R[\omega] = m_0\omega^2\tau\Gamma_R[\omega] \geq 0 \qquad Y_R[\omega] = \frac{Z_R[\omega]}{|Z[\omega]|^2} \geq 0$$

In fact, the impedance and admittance functions are passive functions, which ensures stability of the coupled system [9].

We can then give a simpler derivation, in the spirit of Nyquist's derivation [11–13], of the fluctuation-dissipation relations for coupled position. We first rewrite the relation between the commutator ξ_{FF}^{in} and the dissipative part of the motional susceptibility χ_{FF}^{in} (see eqs 12)

$$\xi_{FF}^{\mathrm{in}}[\omega] = \omega Z_R[\omega]$$

Hence, the non commutative character of velocity fluctuations is related to the dissipative part Y_R of the mechanical admittance

$$\xi_{vv}[\omega] = |Y[\omega]|^2\xi_{FF}^{\mathrm{in}}[\omega] = \omega Y_R[\omega]$$

The result for the position's commutator is the same as previously obtained (see eqs 21)

$$\xi_{qq}[\omega] = \frac{\xi_{vv}[\omega]}{\omega^2} = \frac{Y_R[\omega]}{\omega}$$

APPENDIX B: TIME DEPENDENCE OF THE COMMUTATOR

In order to analyze the time dependence of the coupled commutator $\xi_{vq}[\omega]$, we first consider the case of an unbound mirror coupled to vacuum, whose admittance can be written ($\omega_0 = 0$)

$$Y[\omega] = \frac{1}{(\epsilon - i\omega)\,m[\omega]} = \frac{1}{m_0\,(\epsilon - i\omega)} + \Delta Y[\omega]$$

$$\Delta Y[\omega] = \frac{\tau\Gamma[\omega]}{m_0\,(1 + i\omega\tau\Gamma[\omega])}$$

$$Y(t) = \frac{\theta(t)}{m_0} + \Delta Y(t)$$

In this decomposition, the first part is the uncoupled admittance while the second part represents a correction.

One deduces $Y(0^+)$ from the high frequency behaviour of $Y[\omega]$

$$Y(0^+) = \lim_{\omega \to \infty} \left(-i\omega Y[\omega]\right) = \frac{1}{m_\infty}$$

Starting from the value $\frac{1}{m_\infty}$ for $t = 0^+$, $Y(t)$ decreases to the slightly different value $\frac{1}{m_0}$ for $t \to \infty$. In other words, $Y(t)$ is the sum of a constant value $\frac{1}{m_0}$ and of a bump $\Delta Y(t)$, having a height

$$\Delta Y(0^+) = \frac{1}{m_\infty} - \frac{1}{m_0} = \frac{1}{m_0}\frac{\omega_C\tau}{1 - \omega_C\tau} \approx \frac{\omega_C\tau}{m_0}$$

The integral of the bump is given by

$$\Delta Y[0] = \frac{\tau\Gamma_0}{m_0}$$

so that the width of the bump is of the order of $\frac{\Gamma_0}{\omega_C}$.

One deduces from equations (24)

$$\langle [v(t), q(0)] \rangle = -\frac{i\hbar}{m_0} - i\hbar \left(\Delta Y(t) + \Delta Y(-t)\right)$$

The first term is a canonical commutator (see eq. 10) while the second is a bump-shaped correction, having a width of the order of the mean reflexion delay $\frac{1}{\omega_C}$. At short and long times, one finds equations (26a,b)

$$\langle [v(t), q(0)] \rangle = -\frac{i\hbar}{m_\infty} \qquad \text{for } t \ll \frac{1}{\omega_C}$$
$$= -\frac{i\hbar}{m_0} \qquad \text{for } \frac{1}{\omega_C} \ll t$$

For an harmonically suspended mirror, the admittance function Y can be written as a sum of three components rather than two

$$Y[\omega] = \frac{\rho_+}{i\omega_+ - i\omega} + \frac{\rho_-}{i\omega_- - i\omega} + \Delta Y[\omega]$$
$$Y(t) = \theta(t)\left(\rho_+ \exp(-i\omega_+ t) + \rho_- \exp(-i\omega_- t)\right) + \Delta Y(t)$$

The two complex numbers $\omega_\pm$ are the poles of the admittance function $Y[\omega]$, which are close to the suspension eigenfrequencies $\pm\omega_0$

$$\omega_\pm = \pm\overline{\omega}_0 - \frac{i\gamma}{2} \qquad m_0\omega_0^2 = \omega_\pm^2 m[\omega_\pm]$$
$$\rho_\pm = \frac{1}{2m[\omega_\pm] + \omega_\pm m'[\omega_\pm]}$$

We can estimate these quantities in an expansion with respect to the small parameters $\omega_0\tau$ and $\omega_C\tau$. Assuming that ω_0 is much smaller than ω_C, we get at lowest order

$$\gamma \approx \omega_0^2 \tau \Gamma_0$$
$$\overline{\omega}_0^2 \approx \omega_0^2 + \frac{\gamma^2}{4}$$
$$\rho_\pm \approx \frac{1}{m_0\left(2 + 3i\omega_\pm \tau \Gamma_0\right)}$$

One then deduces by straightforward calculations

$$\Delta Y[0] \approx \frac{\tau\Gamma_0}{m_0} \qquad \Delta Y(0^+) \approx \frac{\omega_C\tau}{m_0}$$

so that the shape of the function ΔY is similar to that previously obtained for an unbound mirror.

Consequently, at short and long times, one finds (26a,c)

$$\langle [v(t), q(0)] \rangle = -\frac{i\hbar}{m_\infty} \qquad \text{for } t \ll \frac{1}{\omega_C}$$
$$= -\frac{i\hbar}{m_0}\cos\left(\overline{\omega}_0 t\right)\exp\left(-\frac{\gamma|t|}{2}\right)$$
$$\text{for } \frac{1}{\omega_C} \ll t$$

APPENDIX C: CASE OF AN ANHARMONIC SUSPENSION

For the sake of simplicity, the simple case of an harmonically suspended mirror has been considered throughout the paper. We show here how the results can be generalized to the case of an anharmonic suspension.

Denoting ω_α the various excitation frequencies from the ground state to the upper states, and q_α the corresponding matrix elements of the position operator, one obtains

$$C_{qq}^{\text{in}}(t) = \sum_\alpha |q_\alpha|^2 \exp(-i\omega_\alpha t)$$

Hence

$$\xi_{qq}^{\text{in}}[\omega] = \frac{1}{2\hbar}\sum_\alpha |q_\alpha|^2 2\pi\left(\delta(\omega - \omega_\alpha) - \delta(\omega + \omega_\alpha)\right)$$
$$C_{qq}^{\text{in}}[\omega] = \sum_\alpha |q_\alpha|^2 2\pi\delta(\omega - \omega_\alpha) = 2\hbar\theta(\omega)\xi_{qq}^{\text{in}}[\omega]$$
$$\sigma_{qq}^{\text{in}}[\omega] = \frac{1}{2\hbar}\sum_\alpha |q_\alpha|^2 2\pi\left(\delta(\omega - \omega_\alpha) + \delta(\omega + \omega_\alpha)\right)$$
$$= \varepsilon(\omega)\xi_{qq}^{\text{in}}[\omega]$$

One also defines the susceptibility

$$\chi_{qq}^{\text{in}}[\omega] = \frac{1}{\hbar}\sum_\alpha |q_\alpha|^2 \frac{2\omega_\alpha}{\omega_\alpha^2 + (\epsilon - i\omega)^2}$$
$$\chi_{qq}^{\text{in}}[\omega] - \chi_{qq}^{\text{in}}[-\omega] = 2i\xi_{qq}^{\text{in}}[\omega]$$

The frequency dependence of the susceptibility is less simple than in the harmonic case. However, the fluctuation-dissipation relations remain valid, for coupled as well as uncoupled variables. Also, the following property remains true for a discrete spectrum

11

$$\frac{1}{\chi_{qq}^{\text{in}}[\pm\omega_\alpha]} = 0$$

$$\frac{\xi_{qq}^{\text{in}}[\omega]}{\chi_{qq}^{\text{in}}[\omega]} = 0$$

which allows one to express the coupled fluctuations in terms of the uncoupled force fluctuations only.

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